Turbulent Diffusion with Memories and Intrinsic Shear

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The first part of the present theory is devoted to the derivation of a Fokker-Planck equation. The eddies smaller than the hydrodynamic scale of the diffusion cloud form a diffusivity, while the inhomogeneous, bigger eddies give rise to a nonuniform migratory drift. This introduces an eddy-induced shear which reflects on the large-scale diffusion. The eddy-induced shear does not require the presence of a permanent wind shear and is intrinsic to the diffusion. Secondly, a transport theory of diffusivity is developed by the method of repeated-cascade and is based upon a relaxation of a chain of memories with decreasing information. The cutoff is achieved by a randomization which brings statistical equilibrium. The number of surviving links varies in accordance with the number of subranges composing the spectrum. Since the intrinsitc shear provides a production subrange added to the inertia subrange, the full range of diffusion consists of inertia, composite and shear subranges. The theory predicts a variance $\sigma^2 \sim t^3$, $t^{12/5}$ and t^2 and an eddy diffusivity $K \sim l^{4/3}$, $l^{7/6}\, l$ for the above diffusions. The coefficients are evaluated. Comparison with experiments in the upper atmosphere and oceans is made.

I. Introduction

The diffusion by irregular movements of particles at time t and position $\boldsymbol{x}$ can be described by a scale-independent phenomenological model, or by a scale-dependent dynamical model. For molecular diffusion, the scale-independent diffusivity D yields a mean square displacement of particles, or variance

$$\sigma^2 = 2\,D\,t\,. \tag{1 a}$$

The analysis of correlations equally lead Taylor [1] to a scale-independent eddy diffusion K, so that the variance

$$\sigma^2 = 2\,K\,t \tag{1 b}$$

in a travel time t larger than the duration of correlation took the same form as (1 a).

Observations on diffusion suggested the need of a more structural and therefore scale-dependent diffusivity

$$K(l) \sim l^{4/3}\,, \tag{2 a}$$

as proposed by Richardson [2], giving a variance

$$\sigma^2 \sim t^3\,. \tag{2 b}$$

The arguments did not provide an analytical basis of the scale-dependence. For this reason, Tchen [3] gave a kinetic foundation by incorporating the scale into a two-particle distribution function and the position into a one-particle distribution function. For an isotropic diffusion, that theory reproduced the Heisenberg law of diffusivity [4] similar to (2 a), and a variance similar to (2 b). For a shear diffusion, it predicted a diffusivity

$$K(l) \sim l \tag{3}$$

and a variance

$$\sigma^2 \sim t^2 \tag{4}$$

on the basis of an energy spectrum [5]

$$F \sim k^{-1}\,. \tag{5}$$

The variance (4) agrees with observations in large-scale diffusion. In addition to the above questions on diffusivities, there were discussions on the influence of a wind shear on diffusion in the presence of a nonuniform convection [6, 7]. In view of the importance of diffusion in atmosphere [8–10] and in oceans [7, 11], and of the need for clarifying the above issues, we propose to develop a theory of turbulent diffusion by decomposition into repeated-cascade.

We shall derive a Fokker-Planck equation of turbulent diffusion in Section II, and formulate a memory-chain of eddy transports in Section III. It is to be remarked that the existence of a memory is basic to the Kubo formualism of diffusion [12] and to the derivation of the Fokker-Planck equation from the BBGKY kinetic hierarchy [13]. The method of repeated-cascade has served to close the correlation hierarchy in hydrodynamic [14], plasma [15] and gravity turbulence [16]. Explicit formulas of eddy diffusivity will be derived for inertia, shear and composite diffusions in Section IV. Solutions of the Fokker-Planck equation will be given in Section V, and comparison with experiments will be made in Section VI.

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II. Derivation of the Fokker-Planck Equation for Turbulent Diffusion

The diffusion of particles has always been modeled by a partial differential equation of the parabolic type. It was first examined by Fokker [17, 18] and Planck [19] in quantum theory, and was analyzed by Kolmogoroff [20] in probability theory. By treating the transition probability as characterizing a transport, Tchen [21, 22] rederived the Fokker-Planck equation of diffusion for a Markoff process and found a migratory drift. For a non-Markoff process with a memory due to a long-range force, the transition was replaced by a closure of kinetic hierarchy, and a generalized Fokker-Planck equation was obtained [13].

In the derivation of a Fokker-Planck equation for turbulent diffusion which follows, we shall exploit all the aforementioned features: differential scales, migratory drift and memories.

The density $n(t, \boldsymbol{x})$ of particles in a turbulent medium of velocity $\boldsymbol{u}(t, \boldsymbol{x})$ is governed by the following equations:

$$\partial \tilde{n}/\partial t + \tilde{\boldsymbol{u}} \cdot \nabla \tilde{n} = D \nabla^2 \tilde{n} + \tilde{s}, \tag{6}$$

and

$$\partial \tilde{\boldsymbol{u}}/\partial t + \tilde{\boldsymbol{u}} \cdot \nabla \tilde{\boldsymbol{u}} = -(1/\varrho) \nabla \tilde{p} + \nu \nabla^2 \tilde{\boldsymbol{u}}, \tag{7}$$

where $\tilde{p}$ is the hydrostatic spressure, ϱ is the density of the turbulent medium, ν and D are the molecular viscosity and diffusivity, respectively, and $\tilde{s}$ represents a source. The velocity $\tilde{u}$ satisfies the equation of continuity

$$\nabla \cdot \tilde{\boldsymbol{u}} = 0 \tag{8}$$

for an incompressible fluid. We shall not make a distinction between the velocities of particles and fluid. Such a distinction [23] becomes necessary in other circumstances.

We write the variables

$$\tilde{n} = N + n, \quad \tilde{\boldsymbol{u}} = \boldsymbol{U} + \boldsymbol{u}, \quad \tilde{p} = P + p, \quad \tilde{s} = S + s, \tag{9}$$

into mean parts, N, U, P, S and turbulent fluctuations, n, $\boldsymbol{u}$, p and s. An average, denoted later by a bar and taken over a hydrodynamic scale, helps in separating between a mean quantity and a fluctuation. It is called a Reynolds average. Other types of averages, called rank average, will be discussed in Section III.

The mean and turbulent densities are governed by the dynamical equations

$$\partial N/\partial t + \boldsymbol{U} \cdot \nabla N = D \nabla^2 N - \nabla \cdot \overline{n \boldsymbol{u}} + S \tag{10}$$

and

$$\left[\frac{\partial}{\partial t} + (\boldsymbol{U} + \boldsymbol{u}) \cdot \nabla\right] n - \nabla \cdot \overline{n \boldsymbol{u}} = -\boldsymbol{u} \cdot \nabla N + D \nabla^2 n + s, \tag{11 a}$$

which add up to (6). They govern a macroscopic and a random variable along different scales and have different purposes: (10) decribes the macroscopic evolution of the density $N(t, \boldsymbol{x})$, and (11 a) determines the fluctuation $n(t, \boldsymbol{x})$ and, in a certain sense, can be regarded as a Langevin equation of turbulent transport. The flux $-\overline{n \boldsymbol{u}}$ in (10) represents the statistical effect of small fluctuations n and $\boldsymbol{u}$ on N and therefore possesses the larger hydrodynamic scale of N. The same flux in (11 a) has only a formal role of securing an identity after an average, and has no dynamical role of forming correlations from that equation. The fluctuation s is also not useful in contributing to a correlation $\overline{s \boldsymbol{u}}$, since s and $\boldsymbol{u}$ are statistically independent. The dissipation $D \nabla^2 n$ could in principle cause a molecular damping but will be discarded in a mixing process dominated by turbulence. Under those circumstances, we can simplify (11 a) and write in its approximate Langevin form

$$\frac{\mathrm{d}n}{\mathrm{d}t} \equiv \left[\frac{\partial}{\partial t} + (\boldsymbol{U} + \boldsymbol{u}) \cdot \nabla\right] n \cong -\boldsymbol{u} \cdot \nabla N. \tag{11 b}$$

The right hand side represents a stochastic driving force which regulates the transport from a mean density gradient ∇N to a density fluctuation n through the velocity fluctuation $\boldsymbol{u}$. A formal integration of (11 b) along the Lagrangian path gives the density fluctuation

$$n(t) = -\nabla N \cdot \int_0^t \mathrm{d}t' \, \boldsymbol{u}(t') + n(t=0) \tag{12}$$

and the density flux

$$\overline{n u_j} = -\xi_{js} (\partial N/\partial x_s) \tag{13}$$

with an eddy viscosity

$$\xi_{js} = \int_0^t \mathrm{d}t_1 \, \overline{u_j(t) u_s(t_1)} = \int_0^t \mathrm{d}\tau \, \overline{u_j(0) u_s(\tau)}, \quad \tau = t - t_1. \tag{14}$$

The upper limit of integrations in (12) and (14) is the hydrodynamic time scale t.

The representation of an eddy diffusivity (14) by an integral of the Lagrangian correlation confirms the old considerations by Taylor [1] and Kampé de Fériet [24].

In calculating the fluctuation n by (12), we have written the relatively more stationary $\bigtriangledown N$ outside of the integral. Since the duration of the correlation is small as compared to the hydrodynamic time t of the evolution of $N(t, \boldsymbol{x})$, the upper limit t may be replaced by ∞, and the correlation $\overline{n(0)u_j(t)}$ is negligible. Also we have assumed the quasistationarity of $\boldsymbol{u}$ in the correlation

$$\overline{u_j(t)u_s(t_1)} = \overline{u_j(0)u_s(t-t_1)}\,. \tag{15}$$

Upon substituting (13) into (10), we find the following equation of diffusion

$$\partial N/\partial t + \boldsymbol{U}\cdot\bigtriangledown N = D\bigtriangledown^2 N + \bigtriangledown\cdot(\boldsymbol{\xi}\cdot\bigtriangledown N) + S\,. \tag{16}$$

We write the eddy diffusivity in the form

$$\boldsymbol{\xi} = \breve{\boldsymbol{\xi}} + \mathbf{K}\,, \tag{17}$$

where we distinguish a contribution $\breve{\boldsymbol{\xi}}$ by eddies larger than the hydrodynamic scale, and a contribution $\mathbf{K}$ by smaller eddies. They play two distinct roles of convection and diffusion, respectively, yielding approximately

$$\bigtriangledown\cdot(\boldsymbol{\xi}\cdot\bigtriangledown N) \cong \bigtriangledown\breve{\boldsymbol{\xi}}\cdot\bigtriangledown N + \mathbf{K}\cdot\bigtriangledown\bigtriangledown N\,, \tag{18}$$

so that a substitution of (18) reduces (16) to a Fokker-Planck equation

$$\frac{\partial N}{\partial t} = -U_j\frac{\partial N}{\partial x_j} + (D\,\delta_{js} + K_{js})\frac{\partial^2 N}{\partial x_j\,\partial x_s} + S\,. \tag{19}$$

The Fokker-Planck coefficients are the molecular and turbulent diffusivities, and the velocity of convection

$$\breve{U}_j = U_j - \partial\breve{\xi}_{js}/\partial x_s\,. \tag{20}$$

The convection consists of a permanent wind speed U_j and a migratory drift $-\partial\breve{\xi}_{js}/\partial x_s$ due to the inhomogeneity of the eddies which are larger than the hydrodynamic scale of $N(t, \boldsymbol{x})$. This drift was also found by Tchen [21, 22] in the Markoff diffusion. The gradient

$$-\frac{\partial}{\partial x_i}\frac{\partial\breve{\xi}_{js}}{\partial x_s}$$

of the drift of component j in the x_i direction gives and eddy-induced shear, called intrinsic shear. Thus, the apparent shear $\bigtriangledown\breve{U}$ may still exist even in the absence of a permanent wind shear. This explains the oscillatory shape of the cloud envelop and the meandering movement of the centroid sometimes seen in diffusions.

For the sake of simplicity, we restrict the small-scale diffusivity to its diagonal components, by writing

$$K_{js} \cong K_{\beta\beta}\,\delta_{j\beta}\,\delta_{s\beta}\,, \tag{21}$$

where the index $\beta = 1, 2, 3$ is understood as not being subject to the convention of summation. In addition, we assume that the eddy diffusivity equals the eddy viscosity of the fluid, the structure of which will be investigated in Sections III and IV.

III. Memory-Chain of Decreasing Information as the Basis of a Transport Theory of Eddy Viscosity

The method of repeated-cascade decomposes a velocity fluctuation into a series of ranks or filters

$$\boldsymbol{u} = \boldsymbol{u}^{(0)} + \boldsymbol{u}^{(1)} + \ldots + \boldsymbol{u}^{(\alpha)} + \ldots + \boldsymbol{u}^{(N)} \tag{22}$$

of increasing order of randomness, separated by wavenumbers k° *, k', k'' ... which are independent variables, where the rank N may be as large as desired. An accumulated random rank α is obtained by summing over all ranks equal or higher than α, and is

$$\boldsymbol{u}^{\alpha} = \boldsymbol{u}^{(\alpha)} + \boldsymbol{u}^{(\alpha+1)} + \ldots \tag{23 a}$$

By inference, an accumulated macroscopic rank is

$$\overline{\boldsymbol{u}}^{\alpha} = \boldsymbol{u}^{(0)} + \boldsymbol{u}^{(1)} + \ldots + \boldsymbol{u}^{(\alpha-1)}\,. \tag{23 b}$$

A rank $\boldsymbol{u}^{\alpha}(t, \boldsymbol{x})$ in the $\boldsymbol{x}$ space may have a Fourier transform $\boldsymbol{u}^{\alpha}(t, \boldsymbol{k})$, provided that the truncation within the appropriate wavenumber bounds be understood.

In analogy with (14) we define an eddy viscosity

$$\eta^{\alpha}_{js} = \int_0^{t^{\alpha-1}} \mathrm{d}\tau\,\langle u_j{}^{\alpha}(0)\,u_s{}^{\alpha}(\tau)\rangle^{\alpha}\,, \tag{24}$$

using an ensemble everage $\langle\cdots\rangle^{\alpha}$ of rank α, called rank average, which eliminates the rank α but

* Anm. der Redaktion: Um Mißverständnissen vorzubeugen und Verwechslungen mit anderen Exponenten zu vermeiden [siehe z. B. Gln. (47 a), (47 b)], wurde in der ganzen Arbeit statt des oberen Exponenten „Null" das „Grad"-Zeichen gesetzt.

retains lower ranks intact. After such an average, the quantity (24) becomes of rank value lower than α.

Since the rank average is devised to select any rank in the sequence (22), it can be used to derive the dynamic equations for that rank from the hydrodynamic equations governing $\boldsymbol{u}$. Here we shall simply recall that the degradation of $\boldsymbol{u}^{(\alpha)}$ into small scales, in its contribution to (24), is effected by a Reynolds stress $\langle \boldsymbol{u}^{\alpha+1}\boldsymbol{u}^{\alpha+1}\rangle^{\alpha+1}$ proportional to an eddy viscosity $\eta^{\alpha+1}$. On that basis we see that the evolution of η^{α} in (24) is coupled to $\eta^{\alpha+1}$.

For the purpose of formulating the dynamics of the above degeneration and coupling, we consider the dynamical equation for the rank $\boldsymbol{u}^{(\alpha)}$ in $\boldsymbol{k}$ space, as derived by means of the cascade method mentioned above [14, 15],

$$\left[\frac{\mathrm{d}}{\mathrm{d}t} + (\nu_{js} + \eta_{js}^{\alpha+1})\, k_j\, k_s\right] u_i^{(\alpha)} = f_i^{(\alpha)} . \tag{25}$$

Here $f_i^{(\alpha)}$ is a noise field having no correlation with $u_i^{(\alpha)}$, and $(\nu_{js} + \eta_{js}^{\alpha+1})\, k_j\, k_s$ is a relaxation frequency which offers an absorption to the time development of $u_i^{(\alpha)}$ under a noise field $f_i^{(\alpha)}$.

Both Eqs. (11 b) and (25) aim at determining turbulent transports, and therefore are called Langevin equations. However, they differ in that (11 b) pertains to the calculation of a mean flux which is maintained by a macroscopic density gradient, while (25) describes the internal structure of a transport property associated with a group of eddies which are degraded by a train of smaller and more random ones.

Upon multiplying the Langevin equation (25) by $\boldsymbol{u}^{(\alpha)}$ and averaging, we find the differential equation

$$\left[\frac{\mathrm{d}}{\mathrm{d}t} + (\nu\,\delta_{js} + \eta_{js}^{\alpha+1})\, k_j\, k_s\right] \cdot \langle u_r^{(\alpha)}(t, \boldsymbol{k})\, u_m^{(\alpha)}(t', -\boldsymbol{k})\rangle^{\alpha} = 0 , \tag{26}$$

which is integrated to

$$\langle u_r{}^{\alpha}(t, \boldsymbol{k})\, u_m{}^{\alpha}(t', -\boldsymbol{k})\rangle^{\alpha} = \langle u_r{}^{\alpha}(t, \boldsymbol{k})\, u_m{}^{\alpha}(t, -\boldsymbol{k})\rangle^{\alpha} \cdot \exp\{-(\nu\,\delta_{js} + \eta_{js}^{\alpha+1})\, k_j\, k_s (t - t')\} . \tag{27 a}$$

with the use of

$$\langle u_r^{(\alpha)}(t, \boldsymbol{k})\, u_m^{(\alpha)}(t', -\boldsymbol{k})\rangle^{\alpha} = \langle u_r{}^{\alpha}(t, \boldsymbol{k})\, u_m^{\alpha}(t', -\boldsymbol{k})\rangle^{\alpha} . \tag{27 b}$$

Further integrations, with respect to $\boldsymbol{k}$ and $\tau = t - t'$ give the eddy viscosity of rank α

$$\eta_{rm}^{\alpha} = \int_{-\infty}^{\infty} \mathrm{d}\boldsymbol{k}\, \chi^{\alpha} \langle u_r{}^{\alpha}(t, \boldsymbol{k})\, u_m{}^{\alpha}(t, -\boldsymbol{k})\rangle^{\alpha} \cdot \int_0^{\infty} \mathrm{d}\tau \exp\{-(\nu\,\delta_{js} + \eta_{js}^{\alpha+1})\, k_j\, k_s\, \tau\} \tag{28}$$

$$= 2\, c_1 \int_{k^{\alpha-1}}^{\infty} \mathrm{d}k^{\alpha} \frac{F^{\alpha}(k^{\alpha})}{k_j{}^{\alpha}\, k_s{}^{\alpha}(\nu\,\delta_{js} + \eta_{js}^{\alpha+1})} , \quad \alpha = 1, 2, \ldots, N .$$

An accomodation coefficient $c_1 = 0.2$, which is difficult to determine in view of the uncertain damping kernel in (24), has been reported earlier [14], χ^{α} is a scale of conversion in the Fourier transformation, and F^{α} is a spectral distribution of energy of rank α

$$\tfrac{1}{2} \langle u_r{}^{\alpha}\, u_m{}^{\alpha}\rangle^{\alpha} = \int_{k^{\alpha}}^{\infty} \mathrm{d}k\, F_{rm}^{\alpha}(k) . \tag{29}$$

Equation (28), which couples two eddy viscosities at different ranks, describes a chain of relaxation frequencies, called memory-chain. The simplified form neglecting molecular viscosity is, more specifically,

$$\begin{aligned} \eta_{js}' &= 2\, c_1 \int_{k^0}^{\infty} \mathrm{d}k' \; \frac{F_{js}'(k')}{\omega_{k'}} , \\ \eta_{js}'' &= 2\, c_1 \int_{k'}^{\infty} \mathrm{d}k'' \; \frac{F_{js}''(k'')}{\omega_{k''}} , \quad \ldots \\ \eta_{js}^{\alpha} &= 2\, c_1 \int_{k_{\alpha}{}^{-1}}^{\infty} \mathrm{d}k^{\alpha} \; \frac{F_{js}^{\alpha}(k^{\alpha})}{\omega_{k^{\alpha}}} , \quad \ldots \end{aligned} \tag{30}$$

Parallely, the relaxation frequencies

$$\omega_{k^{\alpha}} \equiv k_j{}^{\alpha}\, k_s{}^{\alpha}\, \eta_{js}^{\alpha+1}, \quad \alpha = 1, 2, \ldots \tag{31}$$

are also coupled in a more or less organized manner.

Following (30), the determination of η'_{js} would require the knowledge of the velocity correlations at all higher ranks and previous times $t' > t'' > \ldots$, according to the definition (24).

The chain (30), which is generated by the differential equation (25), pictures the memory as the tracking of information carried by a trace of time correlations

$$\langle \boldsymbol{u}^{(1)}(0)\, \boldsymbol{u}^{(1)}(\tau)\rangle', \langle \boldsymbol{u}^{(2)}(0)\, \boldsymbol{u}^{(2)}(\tau)\rangle'', \ \ldots$$

back into the past times t^0, t', t'', ... Each filtered correlation, e. g. $\langle \boldsymbol{u}^{(1)}(0)\, \boldsymbol{u}^{(1)}(\tau)\rangle'$, is an association which effectuates a time transfer according to (26), whenever a pattern of energy content $\langle \boldsymbol{u}^{(1)2}\rangle'$ is recognized out of the bath of an uncorrelated noise field $\boldsymbol{f}^{(1)}$ and of correlated, but more random, agitation $\boldsymbol{u}''$. The random movements by $\boldsymbol{u}''$ provide an absorbing property η'' at an absorbing frequency $\omega' \equiv k^2\, \eta''$. The evolution of the filtered motion $\boldsymbol{u}^{(1)}$ and of its pattern $\langle \boldsymbol{u}^{(1)}(0)\, \boldsymbol{u}^{(1)}(\tau)\rangle'$ in respect to the sequence (22) during a time t^0 gives an op-

portunity of experiencing its association (coupling) with $\boldsymbol{u}''$ as an absorbing agent, and, therefore, of also recognizing the latter's composition and pattern $\langle \boldsymbol{u}''(0)\boldsymbol{u}''(\tau)\rangle''$ of lower energy content and of duration t'. This coupling process regresses to the past times

$$t' > t'' > t''' \ldots ,$$

perhaps for ever, as in a system of infinitely sharp perception which could recall all minute and microscopic details. Such a system is indeed deterministic. Alternatively, in a stochastic system, the coupling reaches in the long run such a low energy level that it may be completely succumbed to a large number of small and random agitations it receives, so that the last pattern in the chain becomes totally diffuse, assimilated to and burried in the random and unrecognizable environment which does not avail to further tracking. Then in fact a statistical equilibrium is reached, when we write for the last rank α

$$\omega_{k^2} = \omega^{\alpha}_{\text{random}} \tag{32 a}$$

or

$$\eta^{\alpha+1} = \eta^{\alpha+1}_{\text{random}} , \tag{32 b}$$

cutting the chain (30) off at the last link

$$\eta^{\alpha}_{js} = 2\, c_1 \int_{k^{\alpha-1}}^{\infty} \mathrm{d}k^2 \, \frac{F^{\alpha}_{js}(k^2)}{\omega^{\alpha}_{\text{random}}} . \tag{33}$$

We describe three modes of randomization as follows:

(i) In randomization by thermal agitations, we identify

$$\eta^{\alpha+1}_{\text{random}} = \nu . \tag{34 a}$$

(ii) In randomization by isomerization, we regard the memory-chain as a polymer with isomeric links of uniform spectral structure F.

(iii) In randomization by cascade transfer, we call on the smallest eddies which are still capable of cascade transfer for contributing to diffusion, but which are not more capable of further degradation and tracing into the past, since they are totally assimilated with the inertia eddies of the bath, of absorption coefficient $\eta^{\alpha+1}_{\text{random}}$. Since the structure of the bath eddies which shape that property has an environment of rank α with a property η^2 on its own, the last link of the chain finds itself prescribed by a functional relation of the type

$$\eta^{\alpha+1}_{\text{random}} = \eta^{\alpha+1}_{\text{random}}(\eta^2; k^{\alpha}) . \tag{34 b}$$

The mode (i) of randomization is not useful except in diffusion close to the molecular diffusion. The mode (ii) is most simple, and has been exploited in predicting the Kolmogoroff law of isotropic turbulence [14]. The mode (iii) is more comprehensive and is applicable to all diffusions accompanied by a cascade transfer. In this mode, we incorporate the said mechanism of cascade transfer in the eddy viscosity by writing

$$\eta^{\alpha+1}_{\text{random}} = c^{\alpha+1}_{\eta} (\bar{\varepsilon}^2)^{1/3} (k^2)^{-4/3} \tag{34 c}$$

as an extension of the Richardson law (2 a) now including ranks, where

$$(\bar{\varepsilon}^2)^{1/3} = \nu \overline{J}^{\alpha} , \qquad \overline{J}^2 \equiv \langle (\nabla \bar{\boldsymbol{u}}^2)^2 \rangle^{\alpha} \tag{35}$$

are the environmental dissipation and vorticity function, respectively.

We can count on $\bar{\varepsilon}^2$ to fluctuate at its ranking scale as

$$\bar{\varepsilon}^2 = \eta^2 \overline{J}^{\alpha} \tag{36}$$

following a cascade process and thereby we transform (34 c) into

$$\eta^{\alpha+1}_{\text{random}} = c^{\alpha+1}_{\eta} (\eta^2 \overline{J}^2)^{1/3} (k^2)^{-4/3} . \tag{37}$$

This randomization does not depend on external conditions like ν and

$$\varepsilon = \nu \langle (\nabla \boldsymbol{u})^2 \rangle ,$$

and has the essential property of referring $\eta^{\alpha+1}_{\text{random}}$ to η^2, and thus cutting the memory-chain.

In order to be consistent with (21), we write

$$\eta^{\alpha}_{js} \cong \eta^{\alpha}_{\beta\beta}\, \delta_{j\beta}\, \delta_{s\beta} , \tag{38}$$

and shall omit the index in η^2 for the sake of abbreviation. We then reduce the chain (30) to

$$\eta' = c_2 \int_{k^0}^{\infty} \mathrm{d}k' \, \frac{F'(k')}{k'^2 \eta''} , \tag{39 a}$$

$$\eta^2 = c_2 \int_{k^{\alpha-1}}^{\infty} \mathrm{d}k^2 \, \frac{F^2(k^{\alpha})}{(k^2)^2 \eta^{\alpha+1}_{\text{random}}} , \tag{39 b}$$

with $\eta^{\alpha+1}_{\text{random}}$ determined by (37), and c_2 is found to be

$$c_2 = 2\, c_1/3 . \tag{39 c}$$

For diffusions extending to the inertia, shear and composite subranges, we require a memory-chain of two links, with η' and η'' given by (39 a) and (39 b), respectively. Thus we have

$$\eta'' = c_2 \int_{k'}^{\infty} \mathrm{d}k'' \, \frac{F''(k'')}{k''^2 \eta'''_{\text{random}}} , \tag{40}$$

and

$$\eta'''_{\text{random}} = c'''_{\eta}\,(\eta''\overline{J''})^{1/3}\,k''^{-4/3}. \tag{41}$$

Upon solving tor η'' from the system of Eqs. (40) and (41), we find the expression

$$\eta'' = (c_2/c'''_{\eta})^{3/4}\,\overline{J''}^{-1/4}\Big[\int_{k'}^{\infty} \mathrm{d}k''\,k''^{-2/3}F''\Big]^{3/4}, \tag{42}$$

which, substituted into (39 a), will determine η'. The explicit forms of the solutions for η' will be discussed in Section IV.

IV. Explicit Expressions of Eddy Diffusivities

For the unsteady motion of dilute particles, with a Stokes friction linearly proportional to the velocity of a particle relative to the fluid and a friction of acceleration, we have found [22] that the diffusivity of particles can be approximated by the eddy viscosity of the fluid, i. e.

$$K \cong \eta'. \tag{43}$$

For this reason, we shall provisionally not distinguish between the eddy viscosity and eddy diffusivity in the following. Since they may differ sometimes in accordance with their different damping kernels associated with (24), their numerical coefficients may not be determined too accurately when the above approximation is made.

In modeling the diffusions, we distinguish the following subranges:

(a) In an inertia diffusion, the controlling spectrum at all ranks is uniformly the Kolmogoroff law and is valid for small scales.

(b) In a shear diffusion, the controlling spectrum at all ranks lies uniformly in the production subrange. The diffusion is necessary anisotropic and is of large scales.

(c) In a composite diffusion, the spectrum in the inertia subrange controls the mixing over small distances or time intervals, while the spectrum in the production subrange controls the mixing over large distances or time intervals.

Since a cloud of particles may evolve in time or position, we shall discuss the diffusivities according to their governing spectra in the wavenumber and frequency spaces.

A) Inertia Diffusion

In an inertia diffusion, where the spectrum controlling the eddy diffusivities at all ranks follows uniformly the inertia law, we can drastically simplify the calculation of eddy viscosity by reducing the system (39 a) and (42) into the following approximate formula

$$K = (2\,c_2 \int_{k^0}^{\infty} \mathrm{d}k'\,k'^{-2}F)^{1/2} \tag{44 a}$$

in k space, or

$$K = c_2 \int_{\omega^0}^{\infty} \mathrm{d}\omega'\,(F(\omega')/\omega_k') = c_2 \int_{\omega^0}^{\infty} \mathrm{d}\omega'\,\omega'^{-1}F(\omega') \tag{44 b}$$

in ω° space. The conversion between the two spaces is made by means of the dispersion relation

$$\omega_k' = \omega'. \tag{45}$$

The inertia subrange of the energy spectrum in an isotropic turbulence satisfies the Kolmogoroff law [14] in k and ω spaces:

$$F(k) = A_k\,\varepsilon_0^{2/3}\,k^{-5/3}, \quad \text{with } A_k = 1.6, \tag{46 a}$$

$$F(\omega) = A_\omega\,\varepsilon_0\,\omega^{-2}, \quad \text{with } A_\omega = 3.8, \tag{46 b}$$

where ε_0 is the rate of dissipation in an isotropic turbulence. From (44) and (46) we find the eddy diffusivities

$$K = c_\eta'\,\varepsilon_0^{1/3}\,k^{\circ\,-4/3}, \quad \text{with } c_\eta' = 0.4 \tag{47 a}$$

and

$$K = c_\omega'\,\varepsilon_0\,t^{\circ 2}, \text{with } c_\omega' = 0.007, t^\circ = 2\pi/\omega^\circ. \tag{47 b}$$

In a shear flow of gradient

$$\Gamma \equiv \partial \breve{U}_1/\partial x_3 \tag{48}$$

consisting of a permanent wind shear and an intrinsic shear as discussed in (20), an inertia subrange still subsists following a production subrange. Under that circumstance, the inertia spectrum is

$$F_{\alpha\alpha} = A_{\alpha\alpha}\,\varepsilon_{\alpha\alpha}^{2/3}\,k_\alpha^{-5/3}. \tag{49}$$

Formula (49) has the same power law as (46 a), but takes an anisotropic form, with different rates of dissipations

$$\begin{aligned} \varepsilon_{11} &= (\eta_{33} + \eta^*)\,\Gamma^2 + \varepsilon_0, \\ \varepsilon_{22} &= \varepsilon_0, \\ \varepsilon_{33} &= -\eta^*\,\Gamma^2 + \varepsilon_0. \end{aligned} \tag{50}$$

Here η^* is an eddy viscosity originating from pressure fluctuations. Since

$$\varepsilon_{11} > \varepsilon_{22} > \varepsilon_{33}, \tag{51}$$

we expect to have

$$K_{11} > K_{22} > K_{33} \ . \tag{52}$$

B) Shear Diffusion

The governing spectrum has a spectral law [5, 16]

$$F_{11}(k) = \tfrac{1}{3}\, u_\Gamma{}^2\, k^{-1} \tag{53 a}$$

or

$$F_{11}(\omega) = u_\Gamma{}^2\, \omega^{-1} \tag{53 b}$$

in the production subrange, where u_Γ is a shear-dependent velocity proportional to $(\varepsilon/\Gamma)^{1/2}$. It follows from (44)

$$K_{11} = s_\eta{}'\, u_\Gamma\, k^{\circ -1}, \quad s_\eta{}' = 0.36\,, \tag{54 a}$$

$$K_{11} = s_\omega{}' u_\Gamma{}^2\, t^\circ\,, \quad s_\omega{}' = s_\eta{}'^2\,. \tag{54 b}$$

Other components F_{22} and F_{33} do not possess a production subrange, and therefore sustain their inertia laws (49), with their corresponding diffusivities (47).

C) Composite Diffusion

The composite diffusion is governed by two parameters ε and u_Γ characteristic of both the inertia and production subranges. For the purpose of modeling a composite diffusivity from the chain of memories (39 a) and (42), we note first that they involve the moments

$$\int \mathrm{d}k\, k^m\, F \tag{55}$$

of positive order $m = 2$, and of negative orders $m = -2/3$, and $m = -2$, as contributed by small and big eddies, respectively. Therefore by assigning an inertia spectrum (46 a) to the moment of positive order and a production spectrum (53 a) to the moments of negative orders, we can solve the system of Eqs. (39 a) and (42), and find

$$K \equiv \eta' = c'_{\varepsilon\Gamma}(\varepsilon^{1/3}\, u_\Gamma)^{1/2}\, k^{\circ -7/6} \tag{56 a}$$

in k space, with a numerical coefficient $c'_{\varepsilon\Gamma} = 0.56$, or

$$K = C_\varepsilon{}'(\varepsilon^{1/3}\, u_\Gamma)^{6/5}\, t^{\circ 7/5} \tag{56 b}$$

in t° space, with a numerical coefficient $C_\varepsilon{}' \cong 3 \times 10^{-4}$.

V. Dispersion of Particles from a Source

The eddy diffusivity, as determined by a frequency spectrum, is a function of the time interval t°, see (47 b), (54 b) and (56 b). This is the time interval between the input source at $t - t^\circ$ and the density output at t. Thus we can write the solution of the Fokker-Planck equation (19) in the Fourier transform

$$N(t, \boldsymbol{k}) = \int_0^t \mathrm{d}t^\circ\, S(t - t^\circ\, \boldsymbol{k}) \exp\{ -i\, \boldsymbol{k}\cdot\boldsymbol{U}\, t^\circ - k^2 [D + K(t^\circ)]\, t^\circ \} \tag{57}$$

using the approximation (21) and assuming that $\breve{\boldsymbol{U}}$ is locally homogeneous.

The source may be assumed to be gaussian

$$S(t, \boldsymbol{x}) = \left[\frac{P(t)}{(2\pi b^2)^{3/2}} \right] \exp\{ -\boldsymbol{x}^2 / 2\, b^2 \} \tag{58 a}$$

in $\boldsymbol{x}$ space, or

$$S(t, \boldsymbol{k}) = \left[\frac{P(t)}{(2\pi)^3} \right] \exp\{ -\tfrac{1}{2}\, k^2\, b^2 \} \tag{58 b}$$

in $\boldsymbol{k}$ space, where b is the mean width of the gaussian cloud, and $P(t)$ is the intensity of the source at time t

$$\int_{-\infty}^{\infty} \mathrm{d}\boldsymbol{x}\, S(t, \boldsymbol{x}) = P(t)\,. \tag{59}$$

With the use of (58 b), we reduce (57) to

$$N(t, \boldsymbol{k}) = \frac{1}{(2\pi)^3} \int_0^t \mathrm{d}t^\circ\, P(t - t^\circ) \exp[-i\, \boldsymbol{k}\cdot\breve{\boldsymbol{U}}\, t^\circ - \tfrac{1}{2} k^2\, l^2(t^\circ)], \tag{60}$$

where

$$l^2(t^\circ) = b^2 + 2[D + K(t^\circ)]\, t^\circ\,. \tag{61}$$

While the time evolution of $N(t, \boldsymbol{x})$ can be obtained by a Fourier inversion of (60) which will not be discussed, we shall calculate the variance of N, as defined by

$$\sigma^2(t) = \frac{\int_{-\infty}^{\infty} \mathrm{d}\boldsymbol{x}\, (\boldsymbol{x} - \overline{\boldsymbol{x}})^2\, N(t, \boldsymbol{x})}{\int_{-\infty}^{\infty} \mathrm{d}\boldsymbol{x}\, N(t, \boldsymbol{x})}\,, \tag{62}$$

where $\overline{\boldsymbol{x}}$ is the systematic displacement by the convection $\breve{\boldsymbol{U}}$, and is

$$\overline{\boldsymbol{x}} = \frac{\int_{-\infty}^{\infty} \mathrm{d}\boldsymbol{x}\ \boldsymbol{x}\ N(t, \boldsymbol{x})}{\int_{-\infty}^{\infty} \mathrm{d}\boldsymbol{x}\, N(t, \boldsymbol{x})}\,. \tag{63}$$

In noting that

$$N(t,\boldsymbol{k}) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} d\boldsymbol{x}\, N(t,\boldsymbol{x})\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}, \quad (64)$$

and consequently,

$$\frac{\partial N}{\partial \boldsymbol{k}} = -\frac{i}{(2\pi)^3}\int_{-\infty}^{\infty} d\boldsymbol{x}\,\boldsymbol{x}\, N(t,\boldsymbol{x})\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}, \quad (65)$$

$$\frac{\partial^2 N}{\partial \boldsymbol{k}^2} = -\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} d\boldsymbol{x}\,\boldsymbol{x}^2\, N(t,\boldsymbol{x})\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}},$$

we find that the moments are

$$\int_{-\infty}^{\infty} d\boldsymbol{x}\; N(t,\boldsymbol{x} = (2\pi)^3 N(t,\boldsymbol{k})\Big|_{k=0},$$

$$\int_{-\infty}^{\infty} d\boldsymbol{x}\,\boldsymbol{x}\, N(t,\boldsymbol{x}) = (2\pi)^3 i\frac{\partial N}{\partial \boldsymbol{k}}\Big|_{k=0}, \quad (66)$$

$$\int_{-\infty}^{\infty} d\boldsymbol{x}\,\boldsymbol{x}^2 N(t,\boldsymbol{x}) = -(2\pi)^3\frac{\partial^2 N}{\partial \boldsymbol{k}^2}\Big|_{k=0},$$

and that the variance (62) can be calculated most conveniently from the Fourier form (60). By omitting the details of the calculations, we obtain

$$\sigma^2(t) = \frac{\int_0^t dt^\circ\, P(t-t^\circ) l^2(t^\circ)}{\int_0^{t^0} dt^\circ\, P(t-t^\circ)}. \quad (67)$$

We can distinguish two types of sources.

A) Instantaneous Source, i. e. $P(t) = \text{const}\,\delta(t)$

The formulas (61) and (67) give

$$\sigma^2(t) = l^2(t) = b^2 + 2[D + K(t)]t. \quad (68)$$

Upon substituting for K from (47 b), (54 b) and (56 b), we find from (68) the variances in three regimes of diffusion, as follows:

(a) Inertia diffusion

$$\sigma^2(t) = b^2 + 2Dt + 2c_\omega'\varepsilon t^3 \cong 2c_\omega'\varepsilon t^3. \quad (69\,a)$$

(b) Shear diffusion

$$\sigma^2(t) = b^2 + 2Dt + 2s_\omega' u_\Gamma^2 t^2 \cong 2s_\omega' u_\Gamma^2 t^2. \quad (69\,b)$$

(c) Composite diffusion

$$\sigma^2(t) = b^2 + 2Dt + 2C'_{\varepsilon\Gamma}(\varepsilon^{1/3}u_\Gamma)^{6/5} t^{12/5} \quad (69\,c)$$

$$\cong 2C'_{\varepsilon\Gamma}(\varepsilon^{1/3}u_\Gamma)^{6/5} t^{12/5}. \quad (69\,c)$$

B) Continuous Source, i. e. $P(t) = \text{constant}$

Equation (67) reduces to

$$\sigma^2(t) = (1/t)\int_0^t dt^\circ\, l^2(t^\circ)$$

$$b^2 + Dt + (1/t)\int_0^t dt^\circ\, t^\circ K(t^\circ). \quad (70)$$

Upon substituting for K from (47 b), (54 b) and (56 b) again, we find from (70) the variances in three regimes of diffusion, as follows:

(a) Inertia diffusion

$$\sigma^2(t) = b^2 + Dt + \tfrac{1}{2}c_\omega'\varepsilon t^3 \cong \tfrac{1}{2}c_\omega'\varepsilon t^3. \quad (71\,a)$$

(b) Shear diffusion

$$\sigma^2(t) = b^2 + Dt + \tfrac{2}{3}s_\omega' u_\Gamma^2 t^2 \cong \tfrac{2}{3}s_\omega' u_\Gamma^2 t^2. \quad (71\,b)$$

(c) Composite diffusion

$$\sigma^2(t) = b^2 + Dt + \tfrac{5}{6}C'_{\varepsilon\Gamma}(\varepsilon^{1/3}u_\Gamma)^{6/5} t^{12/5}$$

$$\cong \tfrac{5}{6}C'_{\varepsilon\Gamma}(\varepsilon^{1/3}u_\Gamma)^{6/5} t^{12/5}. \quad (71\,c)$$

The approximations (69) and (71) were made on account of the more effective eddy diffusion. The diffusions from an instantaneous source (69) and a continuous source (71) have the same power laws but differ in numerical coefficients only.

It is to be remarked that in a shear flow, the horizontal diffusion is stronger than the vertical diffusion, because the former is in the shear and composite subranges and the latter is in the inertia subrange.

VI. Discussions of Results and Comparison with Experiments

We summarize the results (47 a), (54 a), (56 a), (69), and (71) from an instaneous or a continuous source as follows:

(a) Inertia diffusion

$$K \sim l^{4/3}, \quad \sigma^2 \sim t^3. \quad (72\,a, b)$$

(b) Composite diffusion

$$K \sim l^{7/6}, \quad \sigma^2 \sim t^{12/5}, \quad (73\,a, b)$$

(c) Shear diffusion

$$K \sim l, \quad \sigma^2 \sim t^2. \quad (74\,a, b)$$

The above three subranges of diffusions are ordered in an increasing sequence of time: In the stage (a) of inertia diffusion, the cloud evolves

within the inertia subrange of the spectrum; in the stage (b) of composite diffusion, the cloud overlaps both the inertia and production subranges, and in the stage (c) of shear diffusion, the large cloud is controlled by the eddies in the production subrange. The eddy diffusivity takes the power laws

$$K \sim l^{4/3},\ l^{7/6},\ l \tag{75}$$

and the time growth of the variance takes the power laws

$$\sigma^2 \sim t^3,\ t^{12/5},\ t^2 \tag{76}$$

for the three diffusions.

When we collect the atmospheric diffusion data from different authors over a large time span, and make a proper reduction of data from balloon-pairs into single particles, we find the existence of all three types of diffusion [8, 10]. We have plotted in Fig. 1 the respective power laws (76).

Experiments on diffusion in the upper atmosphere and deep oceans usually do not include the inertia diffusion (72) in view of the large travel times and sizes involved. Experiments on atmospheric diffusion for large travel times [25, 26] support the shear laws (74 a) and (74 b) of diffusion, see Figs. 2 and 3. Data of Fig. 3 were plotted [26] in the frozen

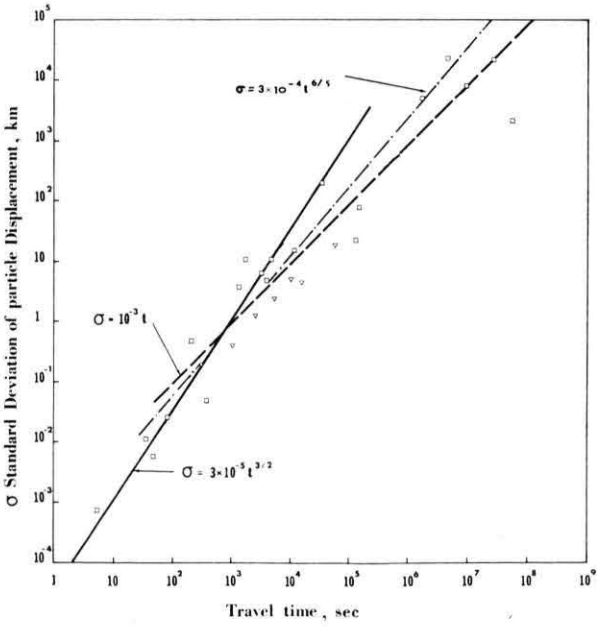

Fig. 1. Standard Deviation of Particles in the Upper Atmosphere Covering All Diffusion Subranges [8, 10]. The power laws $\sigma \sim t^{3/2}$, $t^{6/5}$ and t are theoretical predictions by (76) in inertia, composite and shear diffusions, respectively.

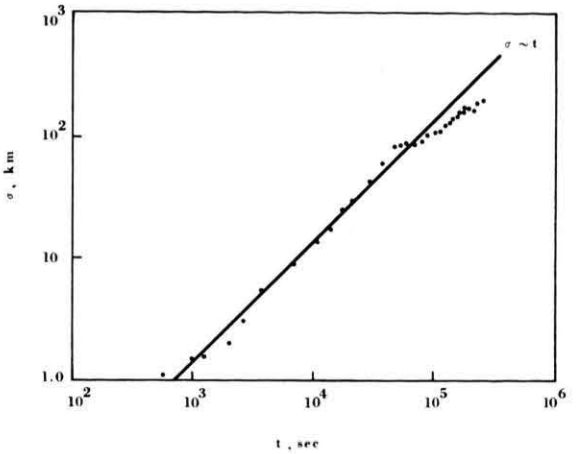

Fig. 2. Standard Deviation of Particles in the Upper Atmosphere for Large Travel Times [25]. The power law $\sigma \sim t$ is the theoretical prediction by (74 b).

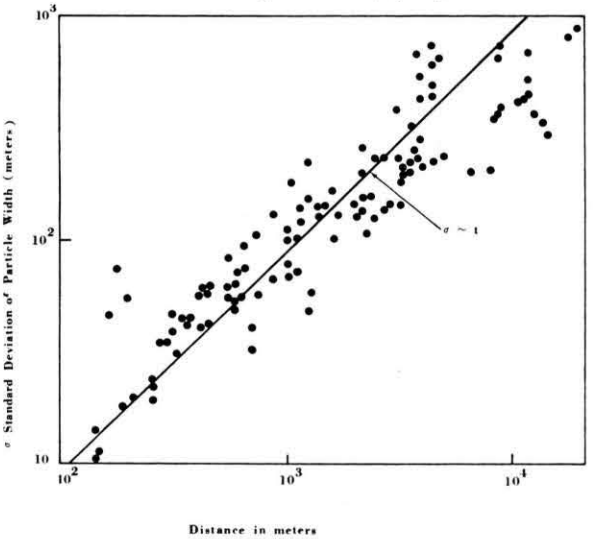

Fig. 3. Standard Deviation of Particles in the Upper Atmosphere for Large Travel Distances [26]. The power law $\sigma \sim t$ is the theoretical prediction by (74 b).

space coordinate which in fact reflects a time coordinate.

Figures 4 and 5 are diffusion measurements in oceans [7, 11]; we have plotted for comparison the power laws

$$K \sim l^{7/6},\ K \sim l,\ \sigma^2 \sim t^{12/5},\ \sigma^2 \sim t^2.$$

The comparison suggests a diffusion of the composite and shear types.

The experimental evidence of the shear law [3] (74 b) was first reported by Zimmerman [27] in support of an earlier theoretical prediction by Tchen [3, 5], and confirmed the numerical modeling by Justus and Hicks [28] and Hicks [29].

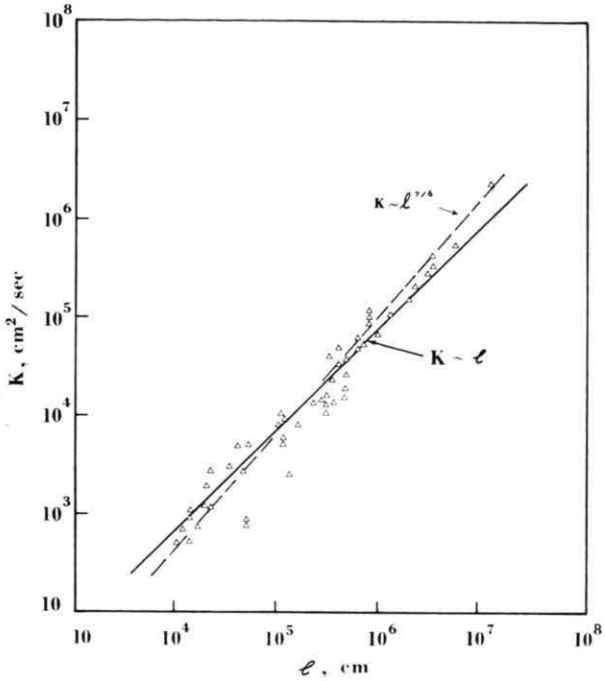

Fig. 4. Scale-Dependence of Eddy Diffusivity in Oceans [7, 11]. The power laws $K \sim l^{7/6}$ and $K \sim l$ are the theoretical predictions by (73 a) and (74 a) in composite and shear diffusions.

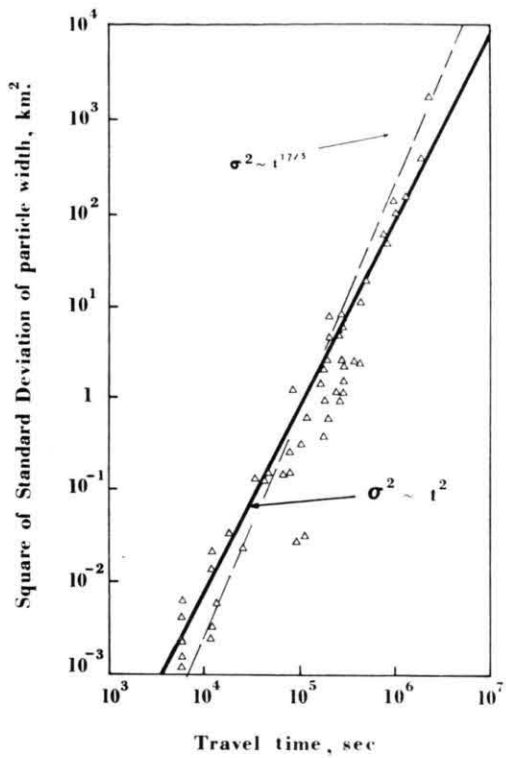

Fig. 5. Variance of Particles in Ocean Diffusion [7, 11]. The power laws $\sigma^2 \sim t^{12/5}$ and t^2 are the theoretical predictions by (73 b) and (74 b) in composite and shear diffusions.

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